Tutorial 3 9.24

① Equality case for the isoperimetric inequality:

C = simple closed curve in $\mathbb{R}^2$ with length l

bdd the interior region Ω with area A

Then $A \leq \frac{l^2}{4\pi}$. Moreover,

"=" holds iff C = a round circle.

Sketch the proof:

$c(s) = (x(s), y(s))$ parametrized by arc-length

Translation and rotation:

$\tilde{c}(s) = (\tilde{x}(s), y(s))$ (may not be parametrized by arc-length)

Green's Thm: $\oint_C P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$P = \frac{-y}{2}, \quad Q = \frac{x}{2}$

$\Rightarrow A = \text{Area}(\Omega) = \int_0^L -\frac{1}{2} y \cdot x' \, ds + \int_0^L \frac{1}{2} x \cdot y' \, ds$

$P = \frac{y}{2}, \quad Q = \frac{x}{2}$

$\Rightarrow -\int_0^L y x' \, ds = \int_0^L x y' \, ds$

$\Rightarrow \ ① \ A = -\int_0^L y x' \, ds$

② $\pi r^2 = \text{Area of the circle } \tilde{C} = \int_0^L \tilde{x}(s) \cdot y'(s) \, ds$

$\overset{①+②}{\Longrightarrow}$

$A + \pi r^2 = \int_0^L \tilde{x}(s) \cdot y'(s) - y(s) \cdot x'(s) \, ds$

$\leq \int_0^L |\langle (\tilde{x}, -y), (y', x') \rangle| \, ds$

$\leq \int_0^L |(\tilde{x}, -y)| \cdot |(y', x')| \, ds$

$= \int_0^L |(\tilde{x}, -y)| \, ds = rL.$

$\sqrt{\pi r^2 A} \leq \frac{A + \pi r^2}{2} \leq \frac{rL}{2} \Rightarrow A \leq \frac{L^2}{4\pi}.$

When the equality holds i.e $A = L^2/4\pi$, all inequalities

above become equalities.

AM-GM ineq. $\Rightarrow$ $A = \pi r^2$

with $A = L^2/4\pi$ $\Rightarrow$ $L = 2\pi r$.

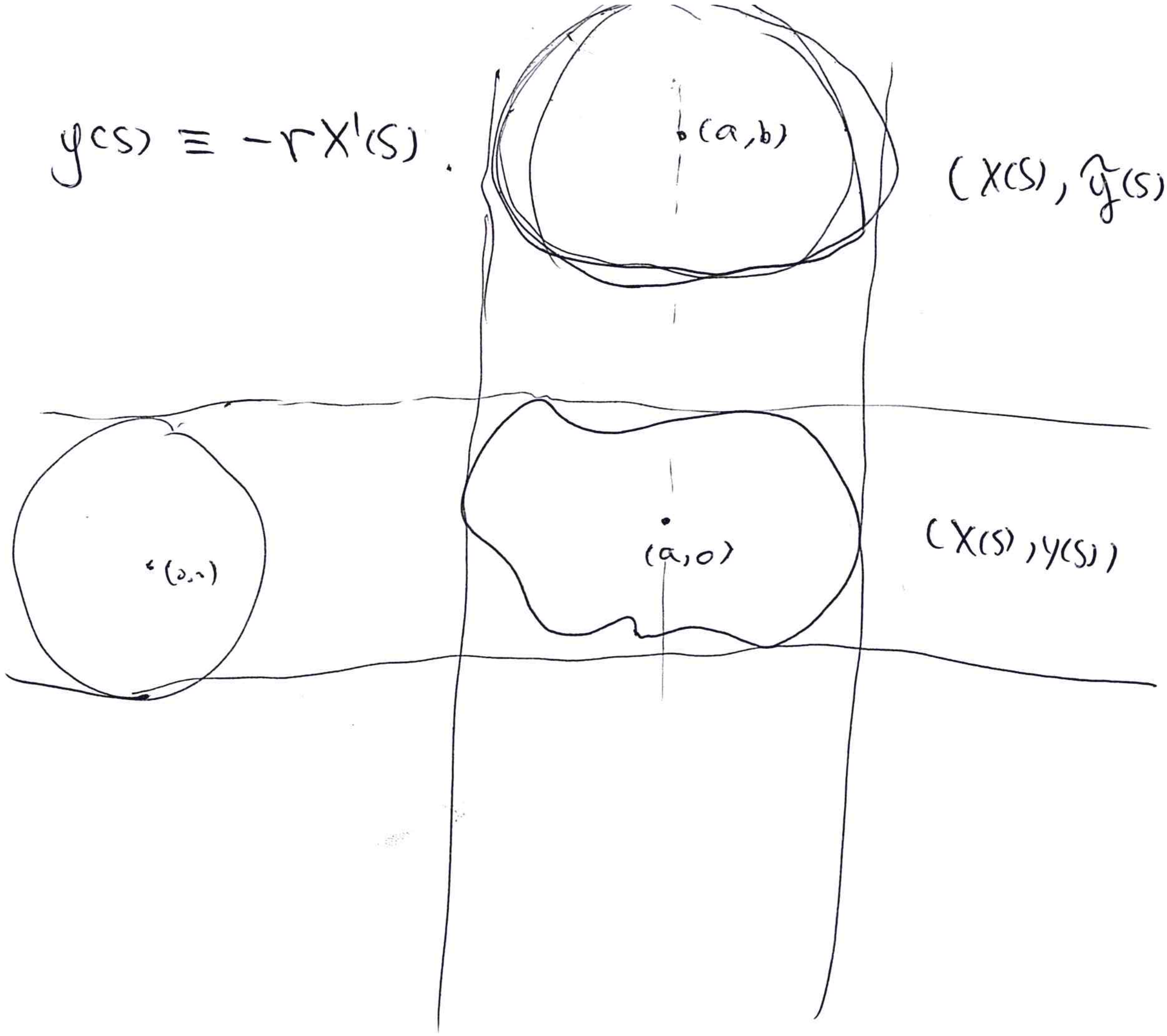
$\Rightarrow$ The width of the slab is indep. of the directions.

C-S ineq. $\Rightarrow$ $\forall s$, $(x(s), -y(s)) = \lambda(s)(y'(s), x'(s))$, $\lambda(s) \geq$

$$\Rightarrow \quad r^2 = \lambda^2(s) \quad \forall s$$

$$\Rightarrow \quad \lambda(s) \equiv r > 0 \quad \forall s$$

$y(s) \equiv -r X'(s).$



~~in the new coordinate, (translate (a,b)→(0,0)~~

$\bar{X}(s) = X(s) - a$

$\bar{y}(s) = \hat{y}(s) - b$

$\underline{y}(s) = y(s) - b$

Do the same thing above.

$A = \int_0^l \bar{x}\,\underline{y}'$ (not strange! $\int_0^l (x-a)y' = \int_0^l xy' - a\int_0^l y'$, $\int_0^l y' ds = y(l) = y(0)$)

$\pi r^2 = -\int_0^l \bar{X}' \bar{y}$

$$A + \pi r^2 = \int_0^l \langle (\bar{x}, -\bar{y}), (\underline{y}', \bar{x}') \rangle$$

$$\leq \int_0^l |\langle (\bar{x}, -\bar{y}), (\underline{y}', \bar{x}') \rangle|$$

$$\leq \int_0^l |(\bar{x}, -\bar{y})|\, ds = rl$$

$$\cdots \quad (\bar{X}(s), -\bar{y}(s)) = \lambda(s)(\underline{y}', \bar{X}')$$

$\Rightarrow \quad \bar{X}(s) \equiv r\underline{y}'$

$\Rightarrow \quad X - a \equiv r y'$

$\Rightarrow \quad (X(s) - a)^2 + (y(s))^2 = r^2(X'^2 + y'^2) = r^2.$

$\Rightarrow \quad C =$ a round circle of radius $r > 0$.

$\square$

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② Theorem on turning tangents (Thm 2.28 in the text-book

$C : [0,1] \to \mathbb{R}^2$ = simple ~~cl~~ closed regular (C^2-) curve

$$\Rightarrow \frac{1}{2\pi}\int_0^1 k(t)\|\dot{C}(t)\|\,dt = \pm 1.$$

Rmk:

$\int_0^1 k(t)\|\dot{c}(t)\|\,dt = \int_0^L k(s)\,ds$ = Total curvature

If $\exists\ C^1\ \varphi$ s.t. ($[0,1] \to \mathbb{R}$).

$e_1(t) = (\cos(\varphi(t)), \sin(\varphi(t)))$

$e_2(t) = (-\sin(\varphi(t)), \cos(\varphi(t)))$

$$k e_2 = \frac{de_1}{ds} = \frac{de_1}{dt}\cdot\frac{dt}{ds} = \dot{\varphi} e_2 \frac{dt}{ds} \Rightarrow k = \frac{d\varphi}{ds} = \frac{d\varphi}{dt}\cdot\frac{dt}{ds} = \frac{\dot{\varphi}}{\|\dot{c}(t)\|}$$

$$\dot\varphi(t) = \kappa(t)\|\dot c(t)\|$$

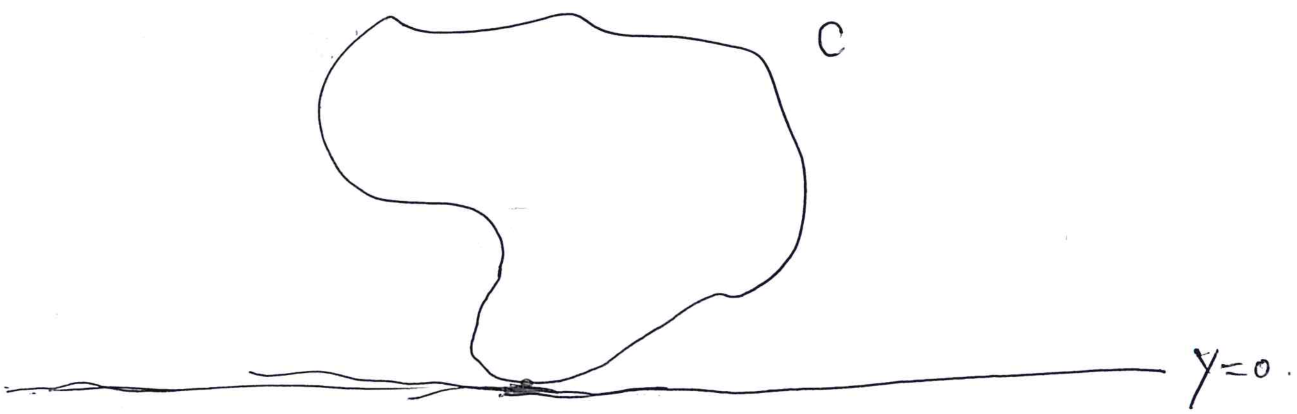
$$\int_0^1 \kappa(t)\|\dot c(t)\|\,dt = \int_0^1 \dot\varphi = \varphi(1) - \varphi(0)$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^1 \kappa(t)\|\dot c(t)\|\,dt = \frac{1}{2\pi}(\varphi(1) - \varphi(0))$$

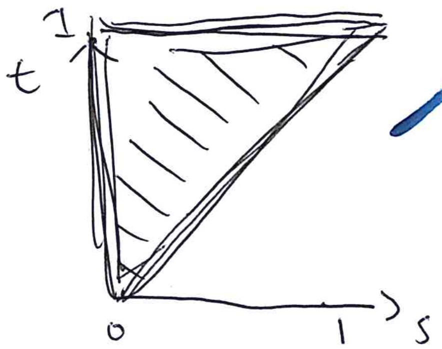
~~thm says~~:

~~c simple closed~~

To show the thm, we need to find such φ.



Pf: $A = \{(s,t) \in \mathbb{R}^2 \mid 0 \le s \le t \le 1\}$



and def: $e: A \to \mathbb{R}^2 \setminus \{0\}$ by

$$e(s,t) = \begin{cases} \dfrac{c(t) - c(s)}{\|c(t) - c(s)\|} & \text{if } s \ne t \text{ and } (s,t) \ne (0,1) \\[2ex] \dfrac{\dot c(t)}{\|\dot c(t)\|} & \text{if } s = t \\[2ex] -\dfrac{\dot c(0)}{\|\dot c(0)\|} & \text{if } (s,t) = (0,1) \end{cases}$$

Simple closed $\Rightarrow$

$$c(t) \neq c(s) \text{ if } t \neq s \text{ and } (s,t) \neq (0,1)$$

$\Rightarrow$ e is well-def.

Check e is cts!

Lemma 2.27 in the text-book:

$e : A \to \mathbb{R}^2 \setminus \{0\}$, A = star-like domain in $\mathbb{R}^2$ with repect to x_0 i.e $\forall x \in A$, $\overline{x x_0} \subset A$

$\Rightarrow$ $\exists$ cts $\varphi : A \to \mathbb{R}$ s.t

$$e(x) = \|e(x)\| \cdot \left(\cos(\varphi(x)), \sin(\varphi(x)) \right).$$

By Lemma 2.27, (taking $x_0 = (0,0)$)

$$e(s,t) = \left(\cos(\varphi(s,t)), \sin(\varphi(s,t)) \right), \quad \varphi(0,0) = 0.$$

$\varphi(t) := \varphi(t,t)$ is what we need!

$$\left(\text{Since } e(t,t) = \frac{\dot{c}(t)}{\|\dot{c}(t)\|} = e_1(t). \right)$$

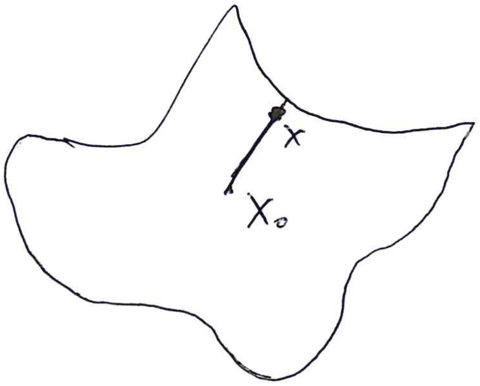
Thus
$$\frac{1}{2\pi} \int_0^1 \kappa(t) \cdot \|\dot{c}(t)\| dt = \frac{1}{2\pi} \left(\varphi(1,1) - \varphi(0,0) \right).$$

But we have

$$\varphi(0,1)-\varphi(0,0)=\pi \quad \text{if } \dot{x}(0)>0 \quad (\text{otherwise } -\pi)$$

$$\varphi(1,1)-\varphi(0,1)=\pi \quad \text{if } \dot{x}(0)>0 \quad (\text{otherwise } -\pi)$$

!/ See the construction of φ in Lemma 2.27:



$e|_{x_0+t(x-x_0)}$, $t\in[0,1]$

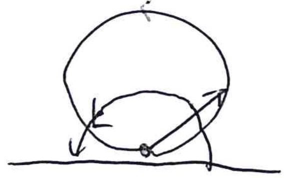
is a cts curve parametrized by t.

If $\varphi(x_0)=0$, then φ is determined uniquely on that curve.

So run x over A, we obtain φ def. on A

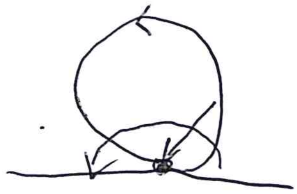
"$\varphi(0,1)-\varphi(0,0)=\pi$"

$\cancel{C(1)}$ $C(t)-C(0)$ $\quad \underset{0\to 1}{t} \quad \Rightarrow \pi.$



"$\varphi(1,1)-\varphi(0,1)=\pi$" $\longrightarrow$ $\boxed{\Rightarrow \varphi(1,1)-\varphi(0,0) = \pm 2\pi}$

$C(1)-C(s)$ $\quad \underset{0\to 1}{s}$



$\Rightarrow \pi.$